

CAUCHY-KOWALEVSKI'S THEOREM APPLIED FOR COUNTING GEOMETRIC STRUCTURES

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ABSTRACT. How many are linear connections with prescribed Ricci tensor? How many are statistical structures? The questions are answered in the analytic case by using the Cauchy-Kowalevski theorem.

1. INTRODUCTION

Our study is inspired by the recent paper of Z. Dušek and O. Kowalski [2]. Roughly speaking, the question is how many structures of a prescribed type there exist. By a satisfactory answer we mean a theorem saying that the set of such structures is parametrized by some family (finite) of arbitrarily chosen functions. We consider the local setting of the question. It turns out that the theorem of Cauchy-Kowalevski can be used as a tool in answering it. Of course, using this tool implies that we must restrict to analytic structures. But the advantage is that the tool belongs to the fundamentals of mathematics and a procedure of getting structures is explicit modulo solving a Cauchy-Kowalevski system of differential equations. On the other hand it seems that the method fits only very special situations.

The paper deals with two questions. The first one is how many connections have a prescribed Ricci tensor. The question of existence of connections with prescribed Ricci tensor was studied, for instance, in [1], [4] and [3]. In particular, it was proved in [4] that any analytic symmetric tensor of type $(0, 2)$ can be locally realized as the symmetric part of the Ricci tensor of some torsion-free connection. We extend this result to not necessarily symmetric prescribed tensors and the whole Ricci tensors. Namely, we observe that a necessary condition for a tensor of type $(0, 2)$ to be (locally) the Ricci tensor of some torsion-free connection is that its anti-symmetric part is a closed form. For an analytic tensor field the closedness of the anti-symmetric part is also a sufficient condition for a local realization as the Ricci tensor of a torsion-free connection. Moreover, we show that the set of all germs at a point in \mathbf{R}^n of analytic torsion-free connections ∇ with prescribed Ricci tensor (whose anti-symmetric part is closed) depends bijectively on $\frac{n^3-3n}{2} + 1$ functions of n variables and $\frac{n^2+n}{2}$ functions of $(n-1)$ variables. In particular, the functions of n variables are some Christoffel symbols of ∇ . Choosing them in special ways one can produce structures with additional properties. In the case of connections with arbitrary torsion we prove, modifying slightly the proof of the main theorem from [2], that the set of all germs of connections with prescribed Ricci tensor depends on $n^3 - n^2$ analytic functions of n variables and n^2 functions on $n-1$ variables.

1991 *Mathematics Subject Classification.* Primary: 35A10, 35Q99, 53B05, 53B20, 35G50.

Key words and phrases. linear connection, Ricci tensor, statistical structure, Cauchy-Kowalevski's theorem.

The first author was supported by the NCN grant UMO-2013/11/B/ST1/02889.

We also consider the case where the trace of the torsion vanishes. We give a partial result to the question how many metric structures there are with prescribed Ricci tensor. Namely, an answer is provided in the 2-dimensional case for non-degenerate Ricci tensors. Here the Cauchy-Kowalevski theorem of the second order is used.

Another question which can be treated by means of Cauchy-Kowalevski's theorem is the one about the amount of statistical structures. A statistical structure is a pair (g, ∇) , where ∇ is a torsion-free connection, g is a metric tensor field and ∇g as a $(0, 3)$ -tensor field is symmetric. Statistical structures are examples of Codazzi pairs. Such structures are very important in differential geometry. For instance, the theory of equiaffine hypersurfaces in \mathbf{R}^n is based on such structures. The theory of the second fundamental form of hypersurfaces in space forms serves as another example. The induced structures of Lagrangian submanifolds in complex space forms are statistical structures. Statistical structures appear in statistics and information geometry. As regards the question we are concerned with, we find how many analytic statistical structures there are around a point in \mathbf{R}^n . The proof of Theorem 4.2 provides an explicit procedure of finding such structures.

2. PRELIMINARIES

Recall the theorem of Cauchy-Kowalevski in the version we need for our considerations. We adopt the notation $(f)_i = \frac{\partial f}{\partial x^i}$, $(f)_{jk} = \frac{\partial^2 f}{\partial x^j \partial x^k}$ for a function on a domain endowed with a coordinate system (x^1, \dots, x^n) . All coordinate systems used in this paper are analytic.

Theorem 2.1. *Consider a system of differential equations for unknown functions U^1, \dots, U^N in a neighborhood of $0 \in \mathbf{R}^n$ and of the form*

$$\begin{aligned} (U^1)_1 &= H^1(x^1, \dots, x^n, U^1, \dots, U^N, (U^1)_2, \dots, (U^1)_n, \dots, (U^N)_2, \dots, (U^N)_n), \\ (U^2)_1 &= H^2(x^1, \dots, x^n, U^1, \dots, U^N, (U^1)_2, \dots, (U^1)_n, \dots, (U^N)_2, \dots, (U^N)_n), \\ &\dots \\ (U^N)_1 &= H^N(x^1, \dots, x^n, U^1, \dots, U^N, (U^1)_2, \dots, (U^1)_n, \dots, (U^N)_2, \dots, (U^N)_n), \end{aligned}$$

where H^i , $i = 1, \dots, N$, are analytic functions of all variables in a neighborhood of $(0, \dots, 0, \varphi^1(0), \dots, \varphi^N(0), (\varphi^1)_2(0), \dots, (\varphi^1)_n(0), \dots, (\varphi^N)_2(0), \dots, (\varphi^N)_n(0)) \in \mathbf{R}^{(N+1)n}$ for analytic functions $\varphi^1, \dots, \varphi^N$ given in a neighborhood of $0 \in \mathbf{R}^{n-1}$.

Then the system has a unique solution $(U^1(x^1, \dots, x^n), \dots, U^N(x^1, \dots, x^n))$ which is analytic around $0 \in \mathbf{R}^n$ and satisfies the initial conditions

$$U^i(0, x^2, \dots, x^n) = \varphi^i(x^2, \dots, x^n) \quad \text{for } i = 1, \dots, N.$$

In the second order Cauchy-Kowalevski theorem we additionally prescribe analytic functions ψ^1, \dots, ψ^N defined in a neighborhood of $0 \in \mathbf{R}^{n-1}$. We have $(U^1)_{11}, \dots, (U^N)_{11}$ on the left-hand sides and we add to the set of arguments of H^1, \dots, H^N on the right-hand sides the first derivatives $(U^1)_1, \dots, (U^N)_1$ and the second derivatives $(U^i)_{jk}$ for $i = 1, \dots, N$, $j = 1, \dots, n$ and $k = 2, \dots, n$. To the initial conditions we add the conditions

$$(U^i)_1(0, x^2, \dots, x^n) = \psi^i(x^2, \dots, x^n)$$

for the prescribed functions ψ^i , $i = 1, \dots, N$.

Since the problems we study are of local nature, we shall locate geometric structures in open neighborhoods of $0 \in \mathbf{R}^n$. For the beginning a neighborhood can be equipped with any analytic coordinate system, for instance, the canonical one.

In the following theorems, when we write about objects in a neighborhood of $0 \in \mathbf{R}^n$, for instance connections, tensor fields, functions, we mean, in fact, their germs at 0.

3. HOW MANY ARE CONNECTIONS WITH PRESCRIBED RICCI TENSOR

For a fixed coordinate system (x^1, \dots, x^n) the Ricci tensor Ric of a linear connection ∇ with Christoffel symbols Γ_{jk}^i is expressed by the formula

$$(1) \quad \text{Ric}(\partial_i, \partial_j) = \sum_{k=1}^n [(\Gamma_{ij}^k)_k - (\Gamma_{kj}^k)_i] + \sum_{k,l=1}^n [\Gamma_{ij}^l \Gamma_{kl}^k - \Gamma_{kj}^l \Gamma_{il}^k].$$

Let r be an analytic tensor field of type $(0, 2)$ around $0 \in \mathbf{R}^n$. Set $r_{ij} = r(\partial_i, \partial_j)$. Modifying arguments from [2] we will prove how many real analytic linear connections ∇ exist such that $\text{Ric} = r$.

The condition $\text{Ric} = r$ is equivalent to the system of equations

$$(2) \quad \sum_{k=1}^n [(\Gamma_{ij}^k)_k - (\Gamma_{kj}^k)_i] = \sum_{k,l=1}^n [\Gamma_{kj}^l \Gamma_{il}^k - \Gamma_{ij}^l \Gamma_{kl}^k] + r_{ij}, \quad i, j = 1, \dots, n.$$

Set

$$(3) \quad \Lambda_{ij} = \sum_{k,l=1}^n [\Gamma_{kj}^l \Gamma_{il}^k - \Gamma_{ij}^l \Gamma_{kl}^k]$$

and rewrite the system (2) in the form

$$(4) \quad [(\Gamma_{ij}^1)_1 + \dots + (\Gamma_{ij}^n)_n] - [(\Gamma_{1j}^1)_i + \dots + (\Gamma_{nj}^1)_i] = \Lambda_{ij} + r_{ij}, \quad i, j = 1, \dots, n.$$

For $i = 1$ and $j = 1, \dots, n$, we keep each derivative $(\Gamma_{nj}^n)_1$ on the left-hand side of the corresponding equation. We denote the sum of all remaining terms on the left-hand side of the corresponding equation by Λ'_{1j} and move it to the right-hand side. For $i > 1$ and $j = 1, \dots, n$, we keep each derivative $(\Gamma_{ij}^1)_1$ on the left-hand side of the corresponding equation. We denote the sum of all remaining terms on the left-hand side of the corresponding equation by Λ'_{ij} and move it to the right-hand side. Then we obtain the (equivalent) system

$$(5) \quad \begin{aligned} (\Gamma_{nj}^n)_1 &= -\Lambda_{1j} - r_{1j} + \Lambda'_{1j}, \quad j = 1, \dots, n, \\ (\Gamma_{ij}^1)_1 &= \Lambda_{ij} + r_{ij} - \Lambda'_{ij}, \quad i = 2, \dots, n, \quad j = 1, \dots, n. \end{aligned}$$

We see that the first derivatives which are on the left-hand sides of this system are not present in any terms on the right-hand sides.

Theorem 3.1. *Let r be an analytic tensor field of type $(0, 2)$ around $0 \in \mathbf{R}^n$. The family of real analytic linear connections ∇ with the Ricci tensor $\text{Ric} = r$ depends bijectively on $n^3 - n^2$ analytic functions of n variables and n^2 analytic functions of $n - 1$ variables.*

Proof. We can choose $n^3 - n^2$ Christoffel symbols Γ_{ij}^k not present on the left hand side of (5) as arbitrary analytic functions. Then n^2 analytic functions of $n - 1$ variables appear by solving the system (5) by the Cauchy-Kowalevski theorem. \square

For a linear connection ∇ with torsion $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$, we have the 1-form τ given by

$$(6) \quad \tau(Y) = \text{tr}(X \rightarrow T(X, Y)) .$$

Using a similar method as above, given an analytic tensor field r around $0 \in \mathbf{R}^n$, we describe all real analytic linear connections Γ such that $\tau = 0$ and $\text{Ric} = r$.

Clearly, this problem is equivalent to finding all solutions of the system consisting of the system (5) and

$$(7) \quad \sum_{i=1}^n (\Gamma_{ik}^i - \Gamma_{ki}^i) = 0, \quad k = 1, \dots, n .$$

Theorem 3.2. *Let $n \geq 3$ and r be an analytic tensor field of type $(0, 2)$ around $0 \in \mathbf{R}^n$. The family of all real analytic linear connections ∇ with $\tau = 0$ and $\text{Ric} = r$ depends bijectively on $n^3 - n^2 - n$ analytic functions of n variables and n^2 analytic functions of $n - 1$ variables.*

Proof. From (7) we have

$$(8) \quad \begin{aligned} \Gamma_{k,k+1}^{k+1} &= -\sum_{i=1}^{k-1} \Gamma_{ki}^i - \sum_{i=k+2}^n \Gamma_{ki}^i \\ &\quad + \sum_{i=1}^{k-1} \Gamma_{ik}^i + \sum_{i=k+1}^n \Gamma_{ik}^i, \quad k = 1, \dots, n-1, \\ \Gamma_{n,n-1}^{n-1} &= -\sum_{i=1}^{n-2} \Gamma_{ni}^i + \sum_{i=1}^{n-1} \Gamma_{in}^i. \end{aligned}$$

Since $n \geq 3$, the Christoffel symbols on the left-hand sides of (8) are not present on the left-hand sides of the n^2 equalities of (5). We substitute the above n equalities (8) into the n^2 equalities of (5). We obtain

$$(9) \quad \begin{aligned} (\Gamma_{nj}^n)_1 &= -\tilde{\Lambda}_{1j} - r_{ij} + \tilde{\Lambda}'_{1j}, \quad j = 1, \dots, n, \\ (\Gamma_{ij}^1)_1 &= \tilde{\Lambda}_{ij} + r_{ij} - \tilde{\Lambda}'_{ij}, \quad i = 2, \dots, n, \quad j = 1, \dots, n, \end{aligned}$$

where $\tilde{\Lambda}_{1j}$, $\tilde{\Lambda}'_{1j}$, $\tilde{\Lambda}_{ij}$, $\tilde{\Lambda}'_{ij}$ are Λ_{1j} , Λ'_{1j} , Λ_{ij} , Λ'_{ij} respectively, after the substitutions. It is easy to see that the first derivatives which are on the left-hand sides of the system (9) are not present on the right-hand sides. Now we can choose $n^3 - n^2 - n$ Christoffel symbols Γ_{jk}^i not present on the left hand sides of (9) and of (8) as arbitrary analytic functions. Then n^2 analytic functions of $n - 1$ variables appear by solving (9) by means of the Cauchy-Kowalevski theorem. \square

If $n = 2$ then the condition $\tau = 0$ yields $T = 0$. Hence the connection is torsion-free. We shall now study this case. Set

$$(10) \quad D_j = \text{div}^\nabla \partial_j = \text{tr}(X \rightarrow \nabla_X \partial_j) = \sum_{k=1}^n \Gamma_{kj}^k.$$

Then the formula for the Ricci tensor can be written as follows

$$(11) \quad \text{Ric}(\partial_i, \partial_j) = \sum_{k=1}^n (\Gamma_{ij}^k)_k - (D_j)_i + \Lambda_{ij}.$$

We decompose the Ricci tensor into its symmetric and anti-symmetric parts, that is, $\text{Ric} = s + a$, where

$$(12) \quad s(X, Y) = \frac{\text{Ric}(X, Y) + \text{Ric}(Y, X)}{2}, \quad a(X, Y) = \frac{\text{Ric}(X, Y) - \text{Ric}(Y, X)}{2}.$$

For a torsion-free connections the portions $\sum_{k=1}^n (\Gamma_{ij}^k)_k$ and Λ_{ij} are symmetric for i and j . Hence for a torsion-free connection we have

$$(13) \quad a_{ij} = a(\partial_i, \partial_j) = \frac{(D_i)_j - (D_j)_i}{2},$$

$$(14) \quad s_{ij} = s(\partial_i, \partial_j) = \sum_{k=1}^n (\Gamma_{ij}^k)_k - \frac{(D_j)_i + (D_i)_j}{2} + \Lambda_{ij}.$$

In [5] the following proposition was proved. Since its proof is short, we cite it here.

Proposition 3.3. *For a torsion-free connection on a paracompact manifold M the anti-symmetric part of its Ricci tensor is exact.*

Proof. By the first Bianchi identity we have

$$\text{tr } R(X, Y) = \text{Ric}(Y, X) - \text{Ric}(X, Y)$$

for a torsion-free connection ∇ , where R is its curvature tensor. Let ∇' be any torsion-free connection whose Ricci tensor Ric' is symmetric. It can be the Levi-Civita connection of some metric. Denote by Q the difference tensor between ∇ and ∇' , that is, $Q(X, Y) = Q_X Y = \nabla_X Y - \nabla'_X Y$. Define the 1-form δ on M by

$$\delta(X) = \text{tr } Q_X.$$

Then

$$d\delta(X, Y) = \frac{1}{2} \{ \text{tr } \nabla' Q(X, Y, \cdot) - \text{tr } \nabla' Q(Y, X, \cdot) \}.$$

The curvature tensors R and R' for ∇ and ∇' are related by the formula

$$R(X, Y)Z = R'(X, Y)Z + \nabla' Q(X, Y, Z) - \nabla' Q(Y, X, Z) + Q_X Q_Y Z - Q_Y Q_X Z.$$

It follows that $\text{tr } R(X, Y) = \text{tr } R'(X, Y) + 2d\delta(X, Y) = 2d\delta(X, Y)$. \square

Since we study problems of local nature, we replace the exactness of the form in the above theorem by its closedness. We shall prove

Theorem 3.4. *A real analytic tensor field r of type $(0, 2)$ can be locally realized as the Ricci tensor of a torsion-free connection if and only if its anti-symmetric part a , that is, $a(X, Y) = \frac{r(X, Y) - r(Y, X)}{2}$, is closed. For a given tensor field r in a neighborhood of $0 \in \mathbf{R}^n$ satisfying the above conditions the set of all analytic torsion-free connections whose Ricci tensor is r , depends bijectively on $\frac{n^3 - 3n}{2} + 1$ arbitrarily chosen analytic functions of n variables and $\frac{n^2 + n}{2}$ arbitrarily chosen analytic functions of $n - 1$ variables.*

Proof. Let s denote the symmetric part of r . The functions $a_{ij} = a(\partial_i, \partial_j)$, $s_{ij} = s(\partial_i, \partial_j)$ are given. Assume that the form a is closed and r is analytic. It is locally exact, hence around the fixed point 0 there is an analytic 1-form α such that $a = -d\alpha$. The 1-form α is chosen up to one function, that is, α can be replaced by $\alpha + d\phi$ for any function ϕ . Let $\alpha = \alpha_1 dx^1 + \dots + \alpha_n dx^n$. We have $2a_{ij} = -2d\alpha(\partial_i, \partial_j) = (\alpha_i)_j - (\alpha_j)_i$. Suppose that r is the Ricci tensor of some torsion-free connection whose Christoffel symbols Γ_{ij}^k are unknown. Then

$$(15) \quad \frac{(D_i)_j + (D_j)_i}{2} = a_{ij} + (D_j)_i$$

for $i, j = 1, \dots, n$. Set $D_i = \alpha_i$ for $i = 1, \dots, n$. We have already used (13) and from now on the functions D_1, \dots, D_n are given.

All the conditions from (14) must be satisfied. We have

$$s_{11} = \sum_{k=1}^n (\Gamma_{11}^k)_k - (D_1)_1 + \Lambda_{11},$$

hence

$$(\Gamma_{11}^1 + \Gamma_{21}^2 + \dots + \Gamma_{n1}^n)_1 = (\Gamma_{11}^1)_1 + (\Gamma_{11}^2)_2 + \dots + (\Gamma_{11}^n)_n + \Lambda_{11} - s_{11}.$$

We can write it equivalently as

$$(16) \quad (\Gamma_{12}^2)_1 = \sum_{k=2}^n (\Gamma_{11}^k)_k - \sum_{k=3}^n (\Gamma_{k1}^k)_1 + \Lambda_{11} - r_{11}.$$

For $i > 1$ we have

$$s_{1i} = \sum_{k=1}^n (\Gamma_{1i}^k)_k - \frac{(D_i)_1 + (D_1)_i}{2} + \Lambda_{1i}.$$

By using (15) we get

$$(\Gamma_{1i}^1)_1 = -(\Gamma_{1i}^2)_2 - \dots - (\Gamma_{1i}^n)_n - \Lambda_{1i} + a_{i1} + (D_1)_i + s_{i1}.$$

We can write it as follows

$$(17) \quad (\Gamma_{1i}^1)_1 = -(\Gamma_{1i}^2)_2 - \dots - (\Gamma_{1i}^n)_n - \Lambda_{1i} + (D_1)_i + r_{i1}.$$

For i, j , where $1 < i \leq j \leq n$, we have

$$s_{ij} = \sum_{k=1}^n (\Gamma_{ij}^k)_k - \frac{(D_j)_i + (D_i)_j}{2} + \Lambda_{ij},$$

that is,

$$s_{ij} = (\Gamma_{ij}^1)_1 + (\Gamma_{ij}^2)_2 + \dots + (\Gamma_{ij}^n)_n - a_{ij} - (D_j)_i + \Lambda_{ij}.$$

We shall write it as follows

$$(18) \quad (\Gamma_{ij}^1)_1 = -(\Gamma_{ij}^2)_2 - \dots - (\Gamma_{ij}^n)_n - \Lambda_{ij} + (D_j)_i + r_{ij}.$$

Collecting the equations from (16)-(18) we get the following Cauchy-Kowalevski system of $\frac{n(n+1)}{2}$ equations (equivalent to (14))

$$(19) \quad \begin{aligned} (\Gamma_{12}^2)_1 &= \sum_{k=2}^n (\Gamma_{11}^k)_k - \sum_{k=3}^n (\Gamma_{k1}^k)_1 + \Lambda_{11} - r_{11}, \\ (\Gamma_{1i}^1)_1 &= -(\Gamma_{1i}^2)_2 - \dots - (\Gamma_{1i}^n)_n - \Lambda_{1i} + (D_1)_i + r_{i1}, \quad i > 1, \\ (\Gamma_{ij}^1)_1 &= -(\Gamma_{ij}^2)_2 - \dots - (\Gamma_{ij}^n)_n - \Lambda_{ij} + (D_j)_i + r_{ij}, \quad 1 < i \leq j \leq n. \end{aligned}$$

The quantities r_{11} , $(D_1)_i + r_{i1}$, $(D_j)_i + r_{ij}$ are given.

Except for the dependence given by (19) the Christoffel symbols are related by the following system of equations

$$(20) \quad \begin{aligned} D_1 &= \Gamma_{11}^1 + [\Gamma_{21}^2] + \dots + \Gamma_{n1}^n, \\ D_2 &= [\Gamma_{12}^1] + \Gamma_{22}^2 + \dots + \Gamma_{n2}^n, \\ &\vdots \\ D_n &= [\Gamma_{1n}^1] + \Gamma_{2n}^2 + \dots + \Gamma_{nn}^n, \end{aligned}$$

where, by using brackets, we marked the Christoffel symbols from the right-hand side of (20) which appear on the left-hand side of (19). Observe also that on the right-hand sides of (20) there are no Christoffel symbols which repeat because of the symmetry of Γ_{ij}^k in lower indices.

From each of the equations in (20) we want to determine one Christoffel symbol and then substitute it into (19) by the expression obtained from (20). Of course, we should not determine and substitute any marked symbol. Moreover, we have to do it in such a way that, after the substitution into (19), the derivatives from the left-hand side of (19) will not appear on the right-hand side of (19). Therefore, from the first equation of (20) we can only take: $\Gamma_{11}^1 = D_1 - \Gamma_{21}^2 - \dots - \Gamma_{n1}^n$. From the next equations we can take Γ_{kk}^k (but here it is not necessary to do it in this way).

For the modified system (19) (after the substitutions) we can apply the Cauchy-Kowalevski theorem.

We shall now count how many Christoffel symbols can be chosen arbitrarily. Note that all Christoffel symbols for which the upper index is equal to one or two of lower indices are on the right-hand side of (20). We see that from (20) we can choose $n(n-2)$ symbols arbitrarily. Consider now the Christoffel symbols for which the upper index is different than each of the lower indices. Consider first the symbols whose upper index is 1. All of them appear on the left-hand side of (19) so we cannot choose them. Finally consider those Christoffel symbols whose upper index is k , where $1 < k \leq n$, and k is different than any of the lower indices. They do not appear neither on the left-hand side of (19) nor on the right-hand side of (20). All of them can be chosen arbitrarily. There are $(n-1)\frac{(n-1)n}{2} = \frac{(n-1)^2n}{2}$ such symbols. Therefore we can choose $n(n-2) + \frac{(n-1)^2n}{2} = \frac{n^3-3n}{2}$ Christoffel symbols arbitrarily. The function ϕ from the beginning of the proof is also an arbitrarily chosen function of n variables.

□

Remark 3.5. For $n = 2$ we have

$$(21) \quad \begin{aligned} D_1 &= \Gamma_{11}^1 + [\Gamma_{12}^2], \\ D_2 &= [\Gamma_{21}^1] + \Gamma_{22}^2. \end{aligned}$$

None of the Christoffel symbols from the right-hand side of (21) can be chosen arbitrarily (in the above procedure). We have $\frac{n^3-3n}{2} = 1$. The only Christoffel symbol which can be arbitrarily chosen in this case is Γ_{11}^2 . In particular, we can choose it 0 and then the vector field $\nabla_{\partial_1} \partial_1$ is parallel to ∂_1 (but we cannot assume that $\nabla_{\partial_1} \partial_1$ vanishes). For any dimension the functions Γ_{11}^k for $k = 2, \dots, n$ are up to choice. In particular, one can choose them 0, which means that $\nabla_{\partial_1} \partial_1$ is parallel to ∂_1 . But we cannot assume that $\Gamma_{11}^1 = 0$. From the last equation of (19) it is clear that we cannot assume that for some $i > 1$ we have $\Gamma_{ii}^k = 0$ for all indices k , because we cannot choose Γ_{ii}^1 arbitrarily.

We shall now give a partial answer to the question how many Levi-Civita connections are those whose Ricci tensor is a prescribed symmetric tensor r of type $(0, 2)$.

For a metric tensor field g (not necessarily positive definite) the Christoffel symbols of its Levi-Civita connection are given by

$$\Gamma_{ij}^s = \frac{1}{2} \sum_{k=1}^n g^{sk} ((g_{ki})_j + (g_{jk})_i - (g_{ji})_k).$$

where $g_{ij} = g(\partial_i, \partial_j)$ and (g^{sk}) is the inverse matrix of the matrix (g_{ij}) . If $n = 2$ and the matrix (g_{ij}) has a diagonal form in the coordinate system then

$$(22) \quad \begin{aligned} \Gamma_{11}^1 &= \frac{1}{2} g^{11} (g_{11})_1, \quad \Gamma_{11}^2 = -\frac{1}{2} g^{22} (g_{11})_2, \quad \Gamma_{21}^1 = \Gamma_{12}^1 = \frac{1}{2} g^{11} (g_{11})_2, \\ \Gamma_{21}^2 &= \Gamma_{12}^2 = \frac{1}{2} g^{22} (g_{22})_1, \quad \Gamma_{22}^1 = -\frac{1}{2} g^{11} (g_{22})_1, \quad \Gamma_{22}^2 = \frac{1}{2} g^{22} (g_{22})_2, \end{aligned}$$

where $g^{11} = \frac{1}{g_{11}}$ and $g^{22} = \frac{1}{g_{22}}$. The Ricci tensor Ric of the Levi-Civita connection for g satisfies the equality $\text{Ric} = fg$, where f is the sectional curvature of g . Using (22), by a straightforward computation one gets

$$(23) \quad \begin{aligned} f &= -\frac{1}{2} g^{11} g^{22} [(g_{11})_{22} + (g_{22})_{11}] \\ &\quad + \frac{1}{4} g^{11} (g^{22})^2 [(g_{22})_2 (g_{11})_2 + ((g_{22})_1)^2] \\ &\quad + \frac{1}{4} (g^{11})^2 g^{22} [(g_{11})_1 (g_{22})_1 + ((g_{11})_2)^2]. \end{aligned}$$

Note that for an analytic metric tensor field on a 2-dimensional manifold there is an analytic orthogonal coordinate system around each point of the domain of the metric tensor field.

Theorem 3.6. *Let r be an analytic non-degenerate tensor field of type $(0, 2)$ such that its matrix is diagonal in an analytic coordinate system (x_1, x^2) on a neighborhood of $0 \in \mathbf{R}^2$. Then the set of all analytic metric tensor fields such that their Ricci tensors equal to r depends bijectively on arbitrarily chosen pairs (φ, ψ) of analytic functions of one variable with $\varphi(0) \neq 0$.*

Proof. Suppose that g is an analytic metric tensor field around $0 \in \mathbf{R}^2$ such that its Ricci tensor Ric is equal to r . Then $g = hr$ for some analytic map h around $0 \in \mathbf{R}^2$ with $h(0) \neq 0$. By (23) the equality $\text{Ric} = r$ is equivalent to the partial differential equation

$$(24) \quad \begin{aligned} &-\frac{1}{2hr_{22}} [(hr_{11})_{22} + (hr_{22})_{11}] \\ &+ \frac{1}{4(hr_{22})^2} [(hr_{22})_2 (hr_{11})_2 + ((hr_{22})_1)^2] \\ &+ \frac{1}{4h^2 r_{11} r_{22}} [(hr_{11})_1 (hr_{22})_1 + ((hr_{11})_2)^2] = r_{11}. \end{aligned}$$

Applying the Leibniz rule one sees that this equation can be transformed equivalently into the one of the form

$$(h)_{11} = F(h, (h)_1, (h)_2, (h)_{12}, (h)_{22})$$

for some analytic map F . Our theorem now follows from the Cauchy-Kowalevski theorem of order 2, where two analytic functions φ, ψ of one variable are prescribed and the initial conditions are: $h(0, x^2) = \varphi$, $(h)_1(0, x^2) = \psi$. \square

4. HOW MANY ARE STATISTICAL STRUCTURES

Recall that a statistical structure on a manifold M is a pair (g, ∇) , where g is a metric tensor field and ∇ is a torsion-free connection on M satisfying the Codazzi condition saying that ∇g as a cubic form is totally symmetric. We assume that the metric is positive definite. A statistical structure is called trace-free when the volume form ν_g determined by g is parallel relative to ∇ . The trace-free statistical structures correspond to Blaschke structures in affine differential geometry and to minimal submanifolds in the theory of Lagrangian submanifolds. We begin with the 2-dimensional case.

In the following Theorems 4.1 and 4.2 the metric tensor fields are unknowns, but according to the Cauchy-Kowalevski theorem they can be arbitrarily chosen at the point 0. Up to linear isomorphism of \mathbf{R}^n we can assume that the matrix (g_{ij}) at 0 is the identity one. We make this assumption for both Theorems 4.1, 4.2, that is, we assume that for the functions g_{ij} appearing in these theorems $g_{ij}(0) = \delta_{ij}$.

Proposition 4.1. *For any analytic linear connection ∇ in a neighborhood of $0 \in \mathbf{R}^2$ there is an analytic metric tensor field g around 0 such that the cubic form ∇g is symmetric. The set of all such metric tensor fields depends on one function g_{11} of two variables and two functions g_{12} , g_{22} of one variable. If additionally the Ricci tensor of ∇ is symmetric then there is a metric tensor field g such that (g, ∇) is a trace-free statistical structure. The set of such metric tensor fields depends on the two functions g_{12} , g_{22} of one variable.*

Proof. For a metric tensor field $g = g_{ij}$ the $(0, 3)$ -tensor ∇g is symmetric if and only if

$$\nabla g(\partial_1, \partial_2, \partial_1) = \nabla g(\partial_2, \partial_1, \partial_1), \quad \nabla g(\partial_1, \partial_2, \partial_2) = \nabla g(\partial_2, \partial_1, \partial_2).$$

It leads to the following Cauchy-Kowalevski system of differential equations with unknowns g_{12} , g_{22}

$$(25) \quad \begin{aligned} (g_{12})_1 &= (g_{11})_2 + \Gamma_{11}^1 g_{12} + \Gamma_{11}^2 g_{22} - \Gamma_{21}^1 g_{11} - \Gamma_{21}^2 g_{21}, \\ (g_{22})_1 &= (g_{12})_2 + \Gamma_{12}^1 g_{12} + \Gamma_{12}^2 g_{22} - \Gamma_{22}^1 g_{11} - \Gamma_{22}^2 g_{12}. \end{aligned}$$

The function g_{11} can be arbitrary (modulo the assumption made before the theorem). Assume now that the Ricci tensor of ∇ is symmetric. In a neighborhood of 0 there is a volume form ν such that $\nabla \nu = 0$. We want to have $\nu = \nu_g$ (up to some constant c), that is,

$$(26) \quad c\nu(\partial_1, \partial_2)^2 = g_{11}g_{22} - g_{12}^2.$$

Since $g_{22} \neq 0$ at 0, we can determine g_{11} from (26) and make the substitution into (25). After the substitution the system remains solvable with two arbitrarily prescribed (modulo the assumption made before the theorem) functions of one variable. \square

The above consideration cannot be repeated in more dimensional cases. But we have

Theorem 4.2. *The set of all analytic statistical structures (g, ∇) around $0 \in \mathbf{R}^n$, where $n > 2$, depends on $\frac{n^3+6n^2+5n}{6}$ arbitrarily chosen analytic functions of n variables, from which one function is g_{11} and $\frac{n^3+6n^2+5n-6}{6}$ functions are some*

Christoffel symbols of ∇ , and $\frac{n(n+1)}{2} - 1$ arbitrarily chosen analytic functions g_{ij} , for $(ij) \neq (1, 1)$, of $(n - 1)$ variables.

Proof. We shall need the following lemma

Lemma 4.3. *A pair (g, ∇) is a statistical structure if and only if*

$$(27) \quad (\nabla g)(\partial_i, \partial_j, \partial_k) = (\nabla g)(\partial_j, \partial_i, \partial_k)$$

for every $i, j, k = 1, \dots, n$ with $i < j$ and $i \leq k$.

Proof. The symmetry for the last two arguments of ∇g holds because of the symmetry of g . Assume (27) for all i, j, k with $i < j$ and $i \leq k$. Take $i, j, k \in \{1, \dots, n\}$ such that $i < j$ and $k < i$. Hence $k < j$. We now have

$$\begin{aligned} (\nabla g)(\partial_i, \partial_j, \partial_k) &= \nabla g(\partial_i, \partial_k, \partial_j) = (\nabla g)(\partial_k, \partial_i, \partial_j) \\ &= (\nabla g)(\partial_k, \partial_j, \partial_i) = (\nabla g)(\partial_j, \partial_k, \partial_i) = (\nabla g)(\partial_j, \partial_i, \partial_k). \end{aligned}$$

□

Consider first the conditions (27) for the indices $1, k, j$, where $1 \leq k \leq j$, that is,

$$(\nabla g)(\partial_1, \partial_j, \partial_k) = (\nabla g)(\partial_j, \partial_1, \partial_k).$$

The conditions lead to the equations

$$(28) \quad (g_{jk})_1 = (g_{1k})_j + \sum_{l=1}^n g_{jl} \Gamma_{1k}^l - \sum_{l=1}^n g_{1l} \Gamma_{jk}^l.$$

There are $\frac{n(n+1)}{2} - 1$ equations in (28). The portion -1 comes from $(g_{11})_1$. The system (28) will be our Cauchy-Kowalevski system. According to the Cauchy-Kowalevski theorem we can prescribe all functions g_{jk} at 0 (in particular). We choose them such that the matrix $g_{jk}(0)$ is the identity one. The function g_{11} can be chosen arbitrarily modulo the assumption that $g_{11}(0) = 1$.

We now take into account the conditions

$$(29) \quad (\nabla g)(\partial_1, \partial_k, \partial_j) = (\nabla g)(\partial_k, \partial_1, \partial_j)$$

for $1 < k < j$. The conditions are equivalent to the equalities

$$(30) \quad (g_{kj})_1 - \sum_{l=1}^n g_{kl} \Gamma_{1j}^l = (g_{1j})_k - \sum_{l=1}^n g_{1l} \Gamma_{kj}^l.$$

Since we postulate that $g_{jk} = g_{kj}$ and $\Gamma_{kj}^l = \Gamma_{jk}^l$, by using (28) we get the conditions

$$(31) \quad (g_{1k})_j + \sum_{l=1}^n g_{jl} \Gamma_{1k}^l = (g_{1j})_k + \sum_{l=1}^n g_{kl} \Gamma_{1j}^l$$

for $1 < k < j \leq n$. We have $(n - 2) + (n - 3) + \dots + 1 = \frac{(n-1)(n-2)}{2}$ equalities in (31). To each equality of (31) we assign the unique pair (k, j) with $k < j$. The obtained correspondence is a bijection between the set of equalities (31) and the set of pairs (k, j) of integers with $1 < k < j \leq n$. So, we can order the system (31) by the inverse lexicographic ordering in pairs (k, j) , that is, $(k_1, j_1) \leq (k, j)$ if and only if $j < j_1$ or $j = j_1$ and $k \leq k_1$.

The rest of the conditions from (27) deal with $(\nabla g)(\partial_i, \partial_j, \partial_k)$, where all i, j, k are different than 1. Assume first that two of the indices i, j, k are equal. We have the equalities

$$(\nabla g)(\partial_i, \partial_j, \partial_i) = (\nabla g)(\partial_j, \partial_i, \partial_i),$$

where $i = 2, \dots, n$ and $j \in \{2, \dots, n\} \setminus \{i\}$. We have $(n-1)(n-2)$ equalities here. They lead to the conditions

$$(32) \quad (g_{ji})_i - \sum_{l=1}^n g_{jl} \Gamma_{ii}^l = (g_{ii})_j - \sum_{l=1}^n g_{il} \Gamma_{ji}^l$$

for $i = 2, \dots, n$ and $j \in \{2, \dots, n\} \setminus \{i\}$. To each equation from (32) we assign the unique pair (j, i) of indices. The obtained correspondence is a bijection between the set of equalities (32) and the set of pairs (j, i) such that $i = 2, \dots, n$ and $j \in \{2, \dots, n\} \setminus \{i\}$. We order the system (32) by means of the inverse lexicographic ordering in pairs (i, j) .

Consider now (27) for all remaining i, j, k , that is, for i, j, k such that $2 \leq i < j \leq n$ and $k \in \{2, \dots, n\} \setminus \{i, j\}$ and $i \leq k$. In fact $i < k$. Hence the condition (27) gives here

$$\sum_{i=2}^{n-2} (n-i)(n-i-1) = \sum_{l=1}^{n-2} l(l-1) = \frac{n^3 - 6n^2 + 11n - 6}{3}$$

equalities

$$(33) \quad (g_{jk})_i - \sum_{l=1}^n g_{jl} \Gamma_{ik}^l = (g_{ik})_j - \sum_{l=1}^n g_{il} \Gamma_{jk}^l$$

for $2 \leq i < j \leq n$ and $k \in \{2, \dots, n\} \setminus \{i, j\}$ and $i \leq k$. To each equality from (33) we assign the unique triple (i, j, k) of indices. This correspondence is a bijection between the set of equalities in (33) and the set of triples (i, j, k) of integers such that $2 \leq i < j \leq n$, $k \in \{2, \dots, n\} \setminus \{i, j\}$ and $i \leq k$. We can order the equalities in (33) by means of the inverse lexicographic ordering in triples (i, j, k) .

Denote by $(*)$ the ordered system of algebraic equations with unknown Christoffel symbols consisting of the above ordered systems (31), (32) and (33) in the sequence (31), (32), (33). From each equation of the system $(*)$, starting from the first equation and going up to the last equation, we want to determine one Christoffel symbol and substitute it into the Cauchy-Kowalevski system as well as into all next equations from our system $(*)$. At each step of the procedure the equations in our Cauchy-Kowalevski system will change and the algebraic equations will change as well. From the subsystem (31) we shall determine symbols Γ_{1k}^j , from the subsystem (32) we shall determine Γ_{ii}^j and from the last subsystem (33) the symbols Γ_{ik}^j . At each step of the procedure our system of differential equations will remain a Cauchy-Kowalevski system and the coefficient in front of the symbol which will be determined at a consecutive step will be non-zero (in some neighborhood of the point 0), that is, it will be possible to determine this symbol from the equation. To this aim we assumed that the matrix of g at 0 is the identity one. Namely, the system $(*)$ evaluated at 0 is the following

$$(34) \quad \begin{aligned} (g_{1k})_j + \Gamma_{1k}^j &= (g_{1j})_k + \Gamma_{1j}^k & \text{for } 1 < k < j \leq n, \\ (g_{ji})_i - \Gamma_{ii}^j &= (g_{ii})_j - \Gamma_{ji}^i & \text{for } i = 2, \dots, n; j \in \{2, \dots, n\} \setminus \{i\}, \\ (g_{jk})_i - \Gamma_{ik}^j &= (g_{ik})_j - \Gamma_{jk}^i & \text{for } 2 \leq i < j \leq n; k \neq j; i < k. \end{aligned}$$

Each Christoffel symbol which we want to determine from the system (34) appears in the system only once. It is easily seen that we can safely apply the procedure described above to (34) because the coefficient in front of a Christoffel symbol which we want to determine at a certain step is non-zero. Hence it is non-zero around a point 0 (because at each step of our procedure we use only elementary algebraic operations) and we can determine this Christoffel symbol in a neighborhood of 0.

After solving the Cauchy-Kowalevski system one goes back to the algebraic system and going from the last to the first equation one gets a complete set of Christoffel symbols.

We have presented an explicit procedure of solving the system of algebraic equations. But, if one does not want an explicit procedure, one can shortly argue as follows. We have the system of algebraic equations with unknowns being Christoffel symbols and consisting of (31) for $1 < k < j \leq n$, (32) for $i = 2, \dots, n$; $j \in \{2, \dots, n\} \setminus \{i\}$ and (33) for $2 \leq i < j \leq n$; $k \neq j$; $i < k$. The system has the form (34) at 0. The matrix of the coefficients of the system is of maximal rank at 0 and so it is around 0. Hence the system has an analytic solution around the point 0 depending on

$$\frac{n^2(n+1)}{2} - \frac{(n-1)(n-2)}{2} - (n-1)(n-2) - \frac{n^3 - 6n^2 + 11n - 6}{3} = \frac{n^3 + 6n^2 + 5n - 6}{6}$$

arbitrarily chosen analytic parameters. It is seen that the substitution of the solutions into the system (28) does not destroy its property of being a Cauchy-Kowalevski system. \square

Remark 4.4. The above theorem and its proof are also valid for g being pseudo-Riemannian metric tensor fields with a fixed signature (modulo linear isomorphisms, as in the above theorem). In the proof it is sufficient to take the matrix of $g(0)$ in an appropriate form.

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